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1 ODE (Ordinary Differential Equation) Basics

A *(ordinary) differential equation (ODE)* relates the derivatives of a function y to y. In these notes, y will by default be a function of t (representing "time"), and we will take derivatives with respect to t, unless otherwise mentioned (the other common choices are y as a function of x, or x as a function of t). Any function appearing in an ODE will be continuous unless otherwise stated, but usually you can assume that you can take as many derivatives as necessary.

When we say we want to solve an ODE, we need to provide all possible solutions, not just one. Usually, ODEs will have a whole *family* of solutions, since the derivatives present do not "see" constants (just like how $f(x)$ and $f(x) + C$ have the same derivative for any constant C). For example, the most basic ODE $y' = y$ has solutions $y(t) = Ce^t$ for any constant C.

As another example, the ODE

$$y' = y^2$$

has solutions $y(t) = \frac{1}{C-t}$ for any constant C, but it also has a "trivial solution" $y \equiv 0$ (by $y \equiv c$ we mean that the function $y(t) = c$ for all t), which must be written down as well. Sometimes the ODE also comes with an initial condition (or conditions) that our solution must satisfy, which turns it into an *initial-value problem* (IVP). For instance, if we impose the condition $y(0) = 0$ on the above ODE, our solutions are now only $y \equiv 0$ and $y(t) = \frac{1}{1-t}$.

Definition 1.1. The *order* of an ODE (or IVP) is the highest derivative that appears in the equation.

For instance, the ODE $y'' + t^2 y = (y')^5$ has order 2.

We can also have *systems* of ODEs, which are simply multiple ODEs that must be satisfied simultaneously. For example, we could have a system

$$x' = x + y - 2, \quad y' = x^2 y.$$

Definition 1.2. An *equilibrium solution* to an ODE $y' = f(t, y)$ is a constant solution. Similarly, an equilibrium solution so a system of ODEs is a solution in which all solution functions are constants.

Example 1.3. In the above system

$$x' = x + y - 2, \quad y' = x^2 y,$$

the equilibrium solutions are obtained when $x(t)$ and $y(t)$ are constant functions, so $x' = y' = 0$. So we need to solve $0 = x + y - 2$, $0 = x^2 y$ for constants x and y. The equilibrium solutions are $(2, 0)$ and $(0, 2)$.

To graphically represent an ODE $y' = f(t, y)$, we can draw a *slope field* or *direction field*. This is done by drawing a short line segment with slope $f(t, y)$ at each point (t, y). We can then picture a sample solution to the ODE by tracing along the line segments in the slope field. Below is an example taken from the book with the ODE $y' = x + y$:

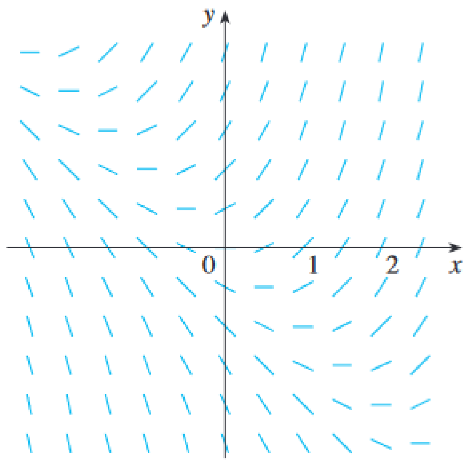

FIGURE 3
Direction field for $y' = x + y$

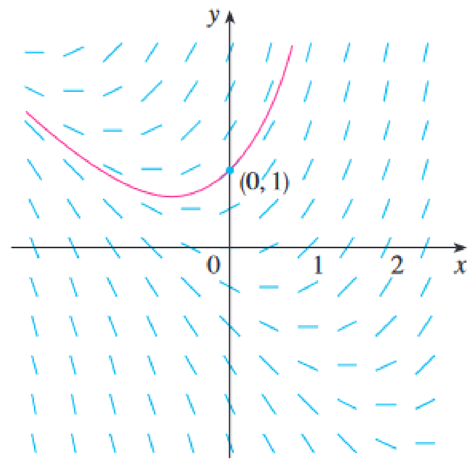

FIGURE 4
The solution curve through (0,1)

There's not much more to say about slope fields.

1.1 Numerical Solutions

We are often not able to exactly solve an ODE. However, *Euler's method* gives us a way to approximate values of the solution function of *first-order* initial value problems, using the idea of linear approximations.

Here is the idea: suppose we have the ODE $y' = f(t, y)$ with initial condition $y(t_0) = y_0$. Fix a *step size* h, which we will use in our linear appoximation. Then for $n \geq 1$, recursively define

$$t_n := t_{n-1} + h, \quad y_n := y_{n-1} + hf(t_{n-1}, y_{n-1}).$$

In other words, to obtain the nth approximation y_n of $y(t_n) = y(t_0 + nh)$ from the $n-1$-st approximation y_{n-1} of $y(t_{n-1})$, we approximate the derivative of y at t_{n-1} by plugging in $f(t_{n-1}, y_{n-1})$. We then use a linear approximation at t_{n-1}: supposing that the function y goes through (t_{n-1}, y_{n-1}) with slope $f(t_{n-1}, y_{n-1})$ at that point, we approximate $y(t_{n-1} + h) = y(t_n)$ to be $y(t_{n-1}) + hf(t_{n-1}, y_{n-1}) \approx y_{n-1} + hf(t_{n-1}, y_{n-1})$. The point is that we use the first-order ODE to iterate a linear approximation over and over.

Example 1.4. Consider the ODE $y' = x + y$ with initial condition $y(0) = 1$. Let's approximate $y(1)$ with step size $h = 0.5$. The approximation for $y(0.5)$ is

$$y(0.5) \approx y_0 + h(t_0 + y_0) = 1 + 0.5 \cdot (0 + 1) = 1.5.$$

So we have $y_1 = 1.5$. Continuing on, the approximation for $y(1)$ is

$$y(1) \approx y_1 + h(t_1 + y_1) = 1.5 + 0.5 \cdot (0.5 + 1.5) = 2.5.$$

2 First-Order ODEs

There are two types of first-order ODEs that you should know how to solve explicitly. The first type is the *separable equation*

$$y' = f(y)g(t), \tag{1}$$

where f and g are continuous functions. In other words, the stuff appearing on the right-hand side of the ODE (after isolating y') can be factored as a function of t and a function of y (where of course y depends on t).

The general method to solve separable ODEs is as follows. Assuming f is never 0, choose a function h of y such that $h'(y) = \frac{1}{f(y)}$ (here the derivative of h is taken with respect to y). Then if $y(t)$ satisfies Equation 1,

$$\frac{d}{dt}h(y(t)) = h'(y(t))y'(t) = \frac{1}{f(y)}f(y)g(t) = g(t),$$

so that

$$h(y(t)) = \int g(t)dt + C \Rightarrow y(t) = h^{-1}\left(\int g(t)dt + C\right).$$

One can check that $y(t)$ of the above form indeed satisfy the ODE (1), so this gives our general solution. If there is an initial condition, that will determine the value of C.

In practice, separation of varables is done using a notational convenience: with Equation 1 as above, we write

$$\frac{dy}{dt} = f(y)g(t) \Rightarrow \frac{1}{f(y)}dy = g(t)dt,$$

integrate both sides (remembering to add in a constant of integation $+C$ after we integrate $g(t)dt$), and then clean up the result. Here is an example:

Example 2.1. Consider the IVP $y' = t(1 + y^2)$, $y(0) = 1$. This is separable, and using our notational shorthand, we can write

$$\frac{1}{1 + y^2}dy = tdt.$$

Integrating both sides, we get $\arctan(y) = t^2/2 + C$, so that the general solution is

$$y(t) = \tan\left(\frac{t^2}{2} + C\right).$$

The initial condition $y(0) = 1$ gives $1 = \tan(C)$, so that $C = \frac{\pi}{4}$ and the solution to the IVP is

$$y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right).$$

Note that C could have been $\frac{\pi}{4} + k\pi$ for any integer k, but this gives the same function $y(t)$ due to the periodicity of tan.

The second type of first-order ODE that you can solve explicitly is the *linear* ODE. This ODE takes the form

$$y' + f(t)y = g(t), \tag{2}$$

where f and g are continuous functions[1]. To solve this, consider the *integrating factor* $I(t) \coloneqq \exp(\int_0^t f(s)ds)$ (the 0 can be replaced with any other number; we really just want an antiderivative of f, and the exponential of any such antiderivative works as an integrating factor). Then notice that if $y(t)$ solves Equation 2, we must have

$$\frac{d}{dt}(I(t)y(t)) = I'(t)y(t) + I(t)y'(t) = f(t)I(t)y(t) + I(t)y'(t) = I(t)g(t),$$

so

$$y(t) = \frac{1}{I(t)}\left(\int_0^t I(s)g(s)ds + C\right).$$

Of course, we need to check that a $y(t)$ of the above form indeed satisfies (2), so the above formula for $y(t)$ really is the general solution.

Example 2.2. We will solve $y' + 2ty = t$. The integrating factor is $\exp(\int 2tdt) = e^{t^2}$, and so multiplying this to both sides, the ODE becomes $e^{t^2}y' + e^{t^2}2ty = te^{t^2}$. The left-hand side is then $(e^{t^2}y)'$, so that

$$e^{t^2}y = \int_0^t se^{s^2}ds = \frac{1}{2}e^{t^2} + C.$$

Hence $y(t) = \frac{1}{2} + Ce^{-t^2}$ is the general solution.

[1] This is called *linear* because it can be written as $L(y) = g(t)$, where L is the *linear differential operator* $y \mapsto \frac{dy}{dt} + fy$. By linear, we mean that L respects addition and scalar multiplication; i.e. that $L(f + g) = L(f) + L(g)$ for functions f and g, and $cL(f) = L(cf)$ for scalars c. You'll learn more about linear maps and operators in Math 54/56/110—these are very very important.

2.1 Applications of first-order ODEs

Let's discuss some applications of first-order ODEs. Consider a family of curves given by $f(x, y, k) = 0$, where f is a fixed function and k varies (e.g. $f(x, y, k) = x^2 + y^2 - k^2$, so the curves are circles with center 0 and radius k). We can ask for an *orthogonal trajectory* to this family, i.e. a single curve that intersects all curves in the family orthogonally. Recall that two curves given implicitly by $(x, c_1(x))$ and $(x, c_2(x))$ intersect orthogonally at (a, b) if $\frac{dc_1}{dx} = -\frac{1}{\frac{dc_2}{dx}}$ at (a, b). That is, the tangent lines of the two curves at (a, b) have slopes that are negative reciprocals of each other.

Example 2.3. We will find an orthogonal trajectory to the family of curves given by $x^2 + 2y^2 = k$. By implicit differentiation, we have $2x + 4y\frac{dy}{dx} = 0$, so that $\frac{dy}{dx} = -\frac{x}{2y}$. Therefore we want to find a function y such that $y'(x) = \frac{2y}{x}$. This is separable, so with $\frac{1}{2y}dy = \frac{1}{x}dx$, integrate both sides and obtain $\frac{\log(y)}{2} = \log(x) + C'$ where C' is the constant of integration. So our orthogonal trajectory is $y = Cx^2$ (for any constant C).

A *mixing problem* is another typical application. In these types of problems, a solution of some substance (usually salt in water) is being poured into a tank of the same substance at some rate, and at the same time the solution in the tank is being continuously mixed and drained at some other rate. The goal is usually to determine how much substance is in the tank at some time t.

The key principles are as follows:

- If x is the amount of substance in the tank, the change $\frac{dx}{dt}$ is equal to the inflow of the substance minus the outflow of the substance.
- The outflow rate of the substance is equal to

$$\frac{\text{Amount of substance in the tank}}{\text{Total volume of solution}} \cdot \text{Outflow rate of solution.}$$

Example 2.4. A very large tank initially contains 100 pounds of salt dissolved in 600 gallons of water. Starting at time $t = 0$, water that contains 0.5 pound of salt per gallon is poured into the tank at a constant rate of 4 gal/min and the (well-mixed) mixture is drained from the tank at a rate of 3 gal/min. We ask how much salt is in the tank after 10 minutes.

Let $x(t)$ be the amount of salt in the tank at time t. The inflow of salt is 2 lb/min, since solution containing 0.5 lb of salt/gallon comes in at the rate of 4 gal/min. The outflow is $\frac{x}{600+t} \cdot 3$, since there is $600 + t$ total gallons of solution after t minutes (as solution flows in at 4 gal/min and out at 3 gal/min), and it drains from the tank at a constant rate of 3 gal/min. Therefore our ODE is

$$\frac{dx}{dt} = 2 - \frac{3x}{600 + t}.$$

This is a linear ODE with integrating factor $\exp\left(\int \frac{3}{600+t}dt\right) = (600+t)^3$, so multiplying this to both sides, the ODE becomes

$$(600+t)^3 x' + 3(600+t)^2 x = 2(600+t)^3.$$

The left-hand side is $((600+t)^3 x)'$, and so we have

$$x(t) = \frac{1}{2}(600+t) + \frac{C}{(600+t)^3}.$$

The initial condition is $x(0) = 100$, so that $C = -43200000000$, and $x(10) \approx 114.6757$.

Finally, there are various population growth and predator-prey models that are given in terms of first-order ODEs. For example, there is a *natural growth model* $P' = kP$ where $k > 0$ is some constant, which represents a population that grows at a rate proportional to its size. The solution is visibly $P(t) = Ce^{kt}$ for initial condition $P(0) = C$. There is also a *logistic growth model* given by the model $P' = kP\left(1 - \frac{P}{M}\right)$, where $k > 0$ as before, and $M > 0$ is a constant representing the *carrying capacity* of the environment (basically the maximum population the environment can sustain, which is a more realistic model). This is a separable ODE, and it is a good exercise to solve it.

As for the predator-prey model, the one you should know is the *Lotka-Volterra model*, which is a system of differential equations

$$\frac{dR}{dt} = kR - aRW, \quad \frac{dW}{dt} = -rW + bRW.$$

Here k, a, r, b are positive constants, R (rabbits) is the prey population, and W (wolves) is the predator population. These equations are more useful for approximate modeling purposes, and cannot be solved (unless you get very lucky with the constants k, a, r, b). On the other hand, you should be able to find the equilibrium solutions of the system (an easy algebra exercise) as well as find $\frac{dW}{dR}$ (given by $\frac{dW}{dt}$ divided by $\frac{dR}{dt}$).

3 Second-Order ODEs

We will exclusively study *linear* second-order ODEs, which have the form

$$a(t)y'' + b(t)y' + c(t)y = g(t), \tag{3}$$

where a, b, c, t are continuous functions of t.

We will distinguish the case when $g(t) \equiv 0$, which is the *homogeneous* case, versus all other *non-homogeneous* cases. Homogeneous second-order linear ODEs have very nice solutions due to the following theorem:

Theorem 3.1. Supposing $a(t)$ is never 0 (so the ODE is always second-order and doesn't degenerate to a lower order), if $y_1(t)$ and $y_2(t)$ are *linearly independent* solutions of

$$a(t)y'' + b(t)y' + c(t)y = 0,$$

then every solution of this ODE is given by some linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ for constants c_1, c_2.

Note that by linearly independent solutions, we mean that $y_1(t)$ and $y_2(t)$ are not scalar multiples (in particular, neither is identically 0, since 0 is a scalar multiple of any function). Also note that this theorem is only true when the ODE is *homogeneous* (why?).

It is still very hard to solve second-order ODEs, so we will focus on the case when $a(t), b(t), c(t)$ are constants. Then in the homogeneous case, the ODE reduces to

$$ay'' + by' + cy = 0. \tag{4}$$

We will always assume that $a \neq 0$.

To solve this, consider the *auxiliary polynomial* $ax^2 + bx + c = 0$. This is a quadratic with two real roots r_1 and r_2 (case 1), a repeated real root r (case 2), or two complex roots $\alpha \pm \beta i$ that are conjugates. Then:

Theorem 3.2. The general solution to Equation 4 is given by:

1. $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$ in case 1.
2. $y(t) = c_1e^{rt} + c_2te^{rt}$ in case 2.
3. $y(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$ in case 3.

Each case of this theorem can be proved by showing that e^{r_1t} and e^{r_2t} (resp. e^{rt} and te^{rt}, $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$) are linearly independent solutions when the auxiliary polynomial satisfies case 1 (resp. case 2, case 3), and then applying Theorem 3.1.

For a second-order ODE, we need to give *two* initial conditions to determine a unique solution, because we take two derivatives. The two initial conditions may take the form of $y(t_0) = y_0$, $y(t_0') = y_0'$ for a fixed time t_0, which gives an *initial value problem*, or the form $y(t_0) = y_0$, $y(t_1) = y_1$ for two different times t_0, t_1, which gives a *boundary value problem* (BVP). These conditions determine the values of c_1 and c_2 much as before (plug in the initial conditions and solve a system of equations if necessary).

We now return to non-homogeneous ODEs, but still with constant coefficients. Now our ODE is of the form

$$ay'' + by' + cy = g(t) \tag{5}$$

with $a \neq 0$. The method to solve this, which is the general heuristic for solving linear ODEs[2], is to first solve the homogenized ODE $ay'' + by' + cy = 0$, obtaining a general solution $y_c(t)$ (remember this is a whole family of solutions). Then, if we can somehow figure out a single *particular solution* $y_p(t)$ to Equation 5, then the general solution is given by $y(t) = y_p(t) + y_c(t)$.

Exercise 3.3. Prove the statement in the previous sentence.

So the difficulty boils down to finding a particular solution y_p. There are two main ways to do this: the method of *undetermined coefficents* and *variation of parameters*.

For the method of undetermined coefficients, the idea is to strategically guess the form of the particular solution y_p, depending on what g looks like. This form will contain coefficients that we can solve for via the given ODE. A rather comprehensive table of guesses to make for y_p (lifted from https://en.wikipedia.org/wiki/Method_of_undetermined_coefficients) is below:

Function of x	Form for y
ke^{ax}	Ce^{ax}
$kx^n,\ n = 0, 1, 2, \ldots$	$\sum_{i=0}^{n} K_i x^i$
$k\cos(ax)$ or $k\sin(ax)$	$K\cos(ax) + M\sin(ax)$
$ke^{ax}\cos(bx)$ or $ke^{ax}\sin(bx)$	$e^{ax}(K\cos(bx) + M\sin(bx))$
$\left(\sum_{i=0}^{n} k_i x^i\right)\cos(bx)$ or $\left(\sum_{i=0}^{n} k_i x^i\right)\sin(bx)$	$\left(\sum_{i=0}^{n} Q_i x^i\right)\cos(bx) + \left(\sum_{i=0}^{n} R_i x^i\right)\sin(bx)$
$\left(\sum_{i=0}^{n} k_i x^i\right)e^{ax}\cos(bx)$ or $\left(\sum_{i=0}^{n} k_i x^i\right)e^{ax}\sin(bx)$	$e^{ax}\left(\left(\sum_{i=0}^{n} Q_i x^i\right)\cos(bx) + \left(\sum_{i=0}^{n} R_i x^i\right)\sin(bx)\right)$

The table should be read as follows: if $g(x)$ (using x instead of t as the independent variable) has the form of a function in the left column, the guess for y_p should have the corresponding form in the right column.

Example 3.4. Let's solve the IVP

$$y'' - 4y' - 12y = te^{4t}, \quad y(0) = 0, \quad y'(0) = 0.$$

[2]This is more of a statement about linear algebra than about differential equations.

The first order of business is to write down the general solution for the homogeneous equation $y'' - 4y' - 12y = 0$. The auxiliary polynomial has roots 6 and -2, so the general solution is $y_c(t) = c_1 e^{6t} + c_2 e^{-2t}$.

It remains to make a guess to find the particular solution y_p. The function $g(t) = te^{4t}$ falls under the last row in the preceding table (with the $\cos(bx)$ term being 1; $b = 0$), so our guess should have the form $(At + B)e^{4t}$. Plugging this guess for y into the ODE, we get

$$e^{4t}(16At + 16B + 8A) - 4e^{4t}(4At + 4B + A) - 12e^{4t}(At + B) = te^{4t}.$$

Cancelling e^{4t} from both sides and simplifying, we get

$$-12At + (4A - 12B) = t.$$

Equating coefficients tells us that $-12A = 1 \Rightarrow A = -\frac{1}{12}$, and $4A - 12B = 0 \Rightarrow B = -\frac{1}{36}$. Therefore a particular solution is $y_p(t) = e^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right)$, and so the general solution is

$$y(t) = y_p(t) + y_c(t) = e^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right) + c_1 e^{6t} + c_2 e^{-2t}.$$

We now need to use the initial conditions. That $y(0) = 0$ tells us that $-\frac{1}{36} + c_1 + c_2 = 0$. Next, $y'(0) = 0$ gives

$$6c_1 - 2c_2 - \frac{7}{36} = 0.$$

We get $a = \frac{1}{32}$ and $b = -\frac{1}{288}$, so the solution to the IVP is

$$y(t) = e^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right) + \frac{1}{32}e^{6t} - \frac{1}{288}e^{-2t}.$$

One caveat with the method of undetermined coefficients occurs in the case where the function we want to guess for our particular solution has terms appearing in the homogeneous general solution $y_c(t)$. In this case, we need to multiply by a sufficiently large power of t in order to make the particular solution independent from anything that appears in the homogeneous solution. As an example, suppose $y_c(t) = c_1 e^t + c_2 te^t$. Then the guess for $y_p(t)$ cannot have the form $(At + B)e^t$, since both Ate^t and Be^t appear as terms in the homogeneous solution. It cannot be of the form $(At^2 + Bt)e^t$ either, since Bte^t appears as a term in the homogeneous solution. On the other hand, going up one degree higher to $(At^3 + Bt^2)e^t$ works as a guess for y_p.

Example 3.5. We will find a particular solution to the ODE

$$y'' + y' - 2y = e^t.$$

Note that the homogeneous equation $y'' + y' - 2y$ has general solution $c_1e^{-2t} + c_2e^t$, so our guess for the form of y_p, which we want to be Ae^t, must in fact be Ate^t upon multiplying by a sufficiently large power of t ($t^1 = t$ works). The point is that Ate^t is (linearly) independent from functions of the form $c_1e^{-2t} + c_2e^t$. Now we can proceed as before to find A: with Ate^t in place of y, we want to solve

$$A(t+2)e^t + A(t+1)e^t - 2Ate^t = e^t,$$

so that $e^t(3A) = e^t$. Therefore $A = \frac{1}{3}$, so a particular solution is $y_p(t) = \frac{te^t}{3}$.

We now discuss the *variation of parameters*. The idea is as follows: using the homogeneous solution $y_c = c_1y_1 + c_2y_2$, where y_1 and y_2 are linearly independent solutions, we suppose that y_p has the form $u_1y_1 + u_2y_2$, where u_1 and u_2 are functions of t that we will determine using our ODE. Then

$$y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2').$$

We now set $u_1'y_1 + u_2'y_2 = 0$ in order to make calculations simpler; the hope is that this condition will always be satisfiable in the end. We then have

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'',$$

and so Equation 5 tells us that we want

$$a(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = g.$$

The left-hand side can be rearranged as

$$u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = g,$$

and since y_1, y_2 solve the homogenized ODE, we conclude that it is equivalent to impose the condition $a(u_1'y_1' + u_2'y_2') = g$.

The upshot is that we need to solve the following system of equations:

$$u_1'y_1 + u_2'y_2 = 0, \quad a(u_1'y_1' + u_2'y_2') = g \tag{6}$$

for *functions* u_1, u_2. It turns out that this system can always be solved for u_1' and u_2'[3]. Once that's done and we integrate u_1' and u_2', we will have obtained a particular solution $y_p = u_1y_1 + u_2y_2$.

[3]Look up the *Wronskian*—the point is that it is always nonzero. You'll need more linear algebra knowledge to appreciate why this is important—see https://math.berkeley.edu/~bsun/docs/Diffeq.pdf.

Example 3.6. We will find a particular solution of the ODE

$$y'' - 2y' + 5y = (t+1)e^t \cos(2t).$$

The general solution to the homogenized equation is $y_c(t) = e^t(c_1 \cos(2t) + c_2 \sin(2t))$, so using the method of undetermined coefficients would be very painful (try it for yourself). We will therefore use variation of parameters with $y_1(t) = e^t \cos(2t)$, $y_2(t) = e^t \sin(2t)$. Skipping directly to the system (6), we want to solve

$$u_1' e^t \cos(2t) + u_2' e^t \sin(2t) = 0, \quad u_1' e^t(\cos(2t) - 2\sin(2t)) + u_2' e^t(\sin(2t) + 2\cos(2t)) = (t+1)e^t \cos(2t).$$

Cancelling e^t everywhere, this is equivalent to

$$u_1' \cos(2t) + u_2' \sin(2t) = 0, \quad u_1'(\cos(2t) - 2\sin(2t)) + u_2'(\sin(2t) + 2\cos(2t)) = (t+1)\cos(2t),$$

and since $u_1' \cos(2t) + u_2' \sin(2t) = 0$, the second equation above reduces to

$$-2u_1' \sin(2t) + 2u_2' \cos(2t) = (t+1)\cos(2t).$$

Multiplying the first equation by $2\sin(2t)$ and the second by $\cos(2t)$ gives the system

$$2u_1' \sin(2t)\cos(2t) + 2u_2' \sin^2(2t) = 0, \quad -2u_1' \sin(2t)\cos(2t) + 2u_2' \cos^2(2t) = (t+1)\cos^2(2t),$$

so adding these equations together gives

$$2u_2' = (t+1)\cos^2(2t).$$

Therefore $u_2' = \frac{t+1}{2}\cos^2(2t)$, and so $u_1' = -\frac{t+1}{2}\sin(2t)\cos(2t)$. Hence a particular solution is

$$y_p(t) = -\left(\int \frac{t+1}{2}\sin(2t)\cos(2t)\right) e^t \cos(2t) + \left(\int \frac{t+1}{2}\cos^2(2t)\right) e^t \sin(2t),$$

where the integrals mean that we just pick any antiderivative of the integrand.

Note that on a test, you should actually carry out and simplify the integrals that arise (but they should be easier to do).

3.1 Applications of second-order ODEs

We also need to discuss some applications of second-order ODEs. These will often come in the context of spring motions. Using some physics knowledge (which I don't really know how to explain), we model the motion of a mass on a *damped* (massless) spring as

$$mx'' + cx' + kx = 0, \tag{7}$$

where $x(t)$ is a function of the position of the spring (with respect to its equilibrium/resting position) in terms of time t, $m > 0$ is the mass of object, $c > 0$ is a damping constant (will be given to you), and $k > 0$ is the *spring constant*. The spring constant k can be calculated by force divided by displacement length (displacement from the resting position). For instance, if a spring has a natural length of 1 meter, and a force of 30N is required to stretch it to a length of 2 meters, the spring constant is $k = \frac{30}{2-1} = 30$.

The cases where the discriminant $c^2 - 4mk$ of the auxiliary polynomial is negative, 0, or positive correspond to the underdamped, critically damped, and overdamped cases respectively.

Example 3.7. Suppose we have a spring with a natural length of 1 meter, and a force of 30N is required to stretch it to a length of 2 meters. Now, we'll attach a 5kg mass to the spring, stretch it to 1.5 meters, and release it with 0 velocity inside a tank of water with damping constant $c = 10$. To set up the second-order IVP modeling this situation, we calculate $k = 30$ as above. Then our ODE is

$$5x'' + 10x' + 30x = 0.$$

The initial conditions are $x(0) = 0.5$, because the spring starts from $1.5 - 1 = 0.5$ meters past its equilibrium position, and $x'(0) = 0$, since the spring is released with 0 velocity (note that the time derivative of the position function $x(t)$ gives velocity). From this information, we can then solve the IVP for the motion of the spring.

We could also add an external force $F(t)$ to the spring, in which case the right-hand side of the ODE (7) becomes $F(t)$:

$$mx'' + cx' + kx = F.$$

All of these ODEs can be solved using previously mentioned techniques.

4 Series Solutions

The final topic of the unit, and of the semester, is about series solutions to ODEs. These are quite useful, because there is often no closed-form formula (in terms of functions you know) to many ODEs. Even very simple-looking ODEs like $y' = \ln(y)$ cannot be solved in elementary terms.

The point of a series solution is that we first *assume* that the solution of an ODE has a series expansion about some initial time $t = t_0$. Plugging in this putative series solution into the ODE, we will get a recursion for the coefficients in our series solutions, which should enable us to solve for the series in terms of the initial coefficients. This should at least give us a solution to our ODE on an interval about t_0, and if we are lucky, the series solution will extend to larger intervals (i.e. the series will have infinite radius of convergence). As proof-of-concept:

Example 4.1. We will find the general solution for the ODE $y' = y$. Of course, we know that the general solutions look like $y(t) = c_0 e^t$ where c_0 is some constant, but let's solve this using series.

Assume we have a series expansion $\sum_{n=0}^{\infty} c_n t^n$ for the solution. Then the condition $y' = y$ forces

$$\sum_{n=0}^{\infty} c_{n+1}(n+1)t^n = \sum_{n=0}^{\infty} c_n t^n,$$

so that $c_{n+1}(n+1) = c_n$ for all $n \geq 0$. It is then easy to see that $c_n = \frac{c_0}{n!}$ for all $n \geq 0$, so that the solution must be

$$\sum_{n=0}^{\infty} \frac{c_0}{n!} t^n = c_0 \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

But we recognize the final series as e^t (valid for all t, not just t near 0), so that a solution must be of the form $c_0 e^t$. Of course, such functions are indeed solutions to the ODE $y' = y$.

Notice, in the above example, how the constant of integration came out from the validity of the recursion not depending on the value of c_0. Of course, if we specified an initial condition for $y(t)$, this would determine the value of c_0, just like before.

Let's do a more involved example, involving an ODE that we don't know how to solve via other techniques.

Example 4.2. We will solve the IVP

$$(t^2 - 1)y'' + 6ty' + 4y = -4, \quad y(0) = 0, \quad y'(0) = 1.$$

Notice that none of our previous methods apply to this ODE. So we will assume that the solution can be written as $y(t) = \sum_{n=0}^{\infty} c_n t^n$. We then have

$$y'' = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n$$

and

$$y' = \sum_{n=0}^{\infty} c_{n+1}(n+1)t^n,$$

so we want

$$\begin{aligned}
-4 &= (t^2-1)\left(\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n\right) + 6t\left(\sum_{n=0}^{\infty} c_{n+1}(n+1)t^n\right) + 4\left(\sum_{n=0}^{\infty} c_n t^n\right) \\
&= \left(\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^{n+2}\right) - \left(\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n\right) \\
&+ \left(\sum_{n=0}^{\infty} 6c_{n+1}(n+1)t^{n+1}\right) + \left(\sum_{n=0}^{\infty} 4c_n t^n\right) \\
&= \left(\sum_{n=2}^{\infty} c_n n(n-1)t^n\right) - \left(\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n\right) + \left(\sum_{n=1}^{\infty} 6c_n n t^n\right) + \left(\sum_{n=0}^{\infty} 4c_n t^n\right) \\
&= -2c_2 - 6c_3 t + 6c_1 t + 4c_0 + 4c_1 t + \sum_{n=2}^{\infty}(c_n(n^2-n) - c_{n+2}(n^2+3n+2) + c_n(6n) + 4c_n)t^n \\
&= (4c_0 - 2c_2) + (10c_1 - 6c_3)t + \sum_{n=2}^{\infty}(c_n(n^2+5n+4) - c_{n+2}(n^2+3n+2))t^n.
\end{aligned}$$

Upon equating coefficients, we find that we need

$$4c_0 - 2c_2 = -4, \quad 10c_1 - 6c_3 = 0,$$

and

$$c_n(n^2+5n+4) - c_{n+2}(n^2+3n+2) = 0$$

for all $n \geq 2$. Since $n^2+5n+4 = (n+1)(n+4)$ and $n^2+3n+2 = (n+2)(n+1)$, it follows that we need $c_n(n+4) - c_{n+2}(n+2) = 0$ for all $n \geq 2$, so we have a recursion

$$c_{n+2} = c_n \frac{n+4}{n+2}.$$

Notice that $c_2 = 2c_0 + 2$, so that $c_4 = \frac{6c_2}{4} = 3c_0 + 3$, $c_6 = \frac{8c_4}{6} = 4c_0 + 4$, and in general $c_{2n} = (n+1)(c_0+1)$ for $n \geq 1$. Similarly, we have $c_3 = \frac{5c_1}{3}$, so that $c_5 = \frac{7c_3}{5} = \frac{7c_1}{3}$, $c_9 = \frac{9c_3}{7} = \frac{9c_1}{3}$, and in general $c_{2n+1} = \frac{(2n+3)c_1}{3}$ for $n \geq 1$. Therefore our general solution about 0 is

$$y(t) = \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_{2n}t^{2n} + \sum_{n=0}^{\infty} c_{2n+1}t^{2n+1} = c_0 + \sum_{n=1}^{\infty}(n+1)(c_0+1)t^{2n} + c_1 t + \sum_{n=1}^{\infty}\frac{(2n+3)c_1}{3}t^{2n+1}.$$

Since $y(0) = c_0$ and $y'(0) = c_1$, it follows that $c_0 = 0$ and $c_1 = 1$, so the IVP solution is

$$y(t) = \sum_{n=1}^{\infty}(n+1)t^{2n} + t + \sum_{n=1}^{\infty}\frac{2n+3}{3}t^{2n+1} = t + 2t^2 + \frac{5}{3}t^3 + 3t^4 + \frac{7}{3}t^5 + 4t^6 + \ldots.$$